Dipartimento di Matematica

A. MARNETTO, L. PANDOLFI

WAVE EQUATIONS, FRACTIONAL DERIVATIVES AND A NEW INSTANCE OF THE LACK OF ROBUSTNESS OF VELOCITY FEEDBACKS

Rapporto interno N. 18, giugno 2007

Politecnico di Torino

Corso Duca degli Abruzzi, 24-10129 Torino-Italia

Wave equations, fractional derivatives and a new instance of the lack of robustness of velocity feedbacks[*]

A. Marnetto[†] L. Pandolfi,[‡]

Abstract

Due to the attenuation of disturbances under integration, it has been proposed that the time derivative of order 1 in the stabilizing feedback of a hyperbolic system be replaced with Caputo fractional derivatives of lower order. It has been already noted that in this way exponential stability is not preserved. We first complement this result. We then prove that even dissipativity is not preserved if the Caputo fractional derivative has to be numerically computed.

key words: wave equations, stabilization, velocity feedback, fractional derivatives

1 Introduction

It is well known that, when $u = 0$, energy is conserved in the following system:

$$w_{tt} = \Delta w \quad \text{in } \Omega \,, \qquad w_{|\Gamma_r} = 0 \,, \quad w_{\nu|\Gamma_a} = -u \tag{1}$$

with initial conditions

$$w(0) = w_0 \,, \qquad w_t(0) = w_1 \,.$$

Notations are as usual: ν is the exterior normal; $\partial\Omega = \Gamma_r \cup \Gamma_a$ and Γ_r (the "reflecting part" of the boundary. $\Gamma_r = \emptyset$ is not excluded) and Γ_a (the

---

[*] This paper fits the research programs of GNAMPA-INDAM

[†] Student, Corso di Laurea in *Ingegneria Matematica,* Politecnico di Torino, Dipartimento di Matematica, Corso Duca degli Abruzzi 24, 10129 Torino, Italy `s140632@studenti.polito.it`

[‡] Politecnico di Torino, Dipartimento di Matematica, Corso Duca degli Abruzzi 24, 10129 Torino, Italy `luciano.pandolfi@polito.it`

"absorbing part" of the boundary) are closed and have empty intersection. Hence, with $u = 0$, the null solution of Eq. (1) is not exponentially stable. Under suitable assumptions on Γ_r and Γ_a, that we don't need to recall here, the feedback

$$u(t) = (w_t)_{|\Gamma_a} \tag{2}$$

exponentially stabilizes system (1), see [3].

When $u(t)$ has to be numerically computed, appearance of small time delays is unavoidable and it is now well understood that arbitrarily small time delays destroy exponential stability of the closed loop. This even in the simplest case $\Omega = (0,1)$, $\Gamma_r = \{1\}$, $\Gamma_a = \{0\}$ (see [1, 4]. See also [8] where it is proved that null controlability is not robust under variations of the delays.)

Due to the fact that integrals attenuates the effects of disturbances, the introduction of Caputo fractional derivatives in the feedback loop has been suggested in [5]; i.e. it has been proposed to replace the feedback condition (2) with

$$u(t) = D_*^\alpha v = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} v'(s)\, \mathrm{d}s\,, \qquad v(t) = w_{|\Gamma_a}\,. \tag{3}$$

Here $\alpha \in (0,1)$.

It turns out that the closed loop system of (1) and (3) is well posed and a state space realization for it has been given in [6]. The details of this realization are not needed. We simply recall that the state space is

$$\mathcal{X} = \mathcal{X}_0 \times L^2(-\infty,+\infty)\,, \qquad \mathcal{X}_0 = \{\phi \in H^1(0,1)\,,\ \phi(1) = 0\} \times L^2(0,1)\,.$$

It is noted in [6] that the closed loop with u given by (3) is strongly stable but not exponentially stable and we can interpret this as a new form of lack of robustness of the velocity feedback since

$$\lim_{\alpha \to 0+} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \phi(s)\, \mathrm{d}s = \phi(t) \tag{4}$$

(even uniformly when $\phi(t)$ is continuous and the domain is bounded, see [9]. For completeness, a simple proof is given in the Appendix). The proof of this negative result as presented in [6] is quite involved since it depends on the special realization proposed there. In this note we first complement this negative result: we give a straightforward proof of the fact that even the component $w(t)$ of the state does not decay exponentially, see Section 2. However, as stated in [5], the solutions remain bounded and the norm of $(w(t;\cdot), w_t(t;\cdot))$ converges to 0 for $t \to +\infty$. Hence the closed loop is not prone to saturation and a natural query is whether the transformation

$$g \to ||(w(t;\cdot), w_t(t;\cdot))||_{\mathcal{X}_0}$$

is continuous, where now g is a perturbation in the feedback loop:

$$u(t) = D_*^\alpha (w_t)_{|\Gamma_a} + g(t)\,. \tag{5}$$

The second goal of this paper is the proof that this transformation *is not* continuous as a transformation from $g \in L^2(0, +\infty)$, to $L^2(0, +\infty; \mathcal{X}_0)$, see Section 3.

So, we remain with the behaviour of the system on a finite time interval. Our final and main goal is the proof of a negative result also in this case. It turns out that the closed loop system of (1) and (3) is dissipative. In fact, the energy of the system is

$$E(t) = \int_0^1 [w_x(t,s)^2 + w_t(t,s)^2]\,\mathrm{d}s$$

and it is known that

$$-E(T) + E(0) = -\frac{1}{2}\int_0^T w_x(t,0)\, w_t(t,0)\,\mathrm{d}t = -\int_0^T u(t)\, v'(t)\,\mathrm{d}t\,. \tag{6}$$

Due to the fact that the Laplace transform of the kernel $1/[\Gamma(1-\alpha)t^\alpha]$ is positive real, it follows that, when $u(t)$ is given by (3), the the right hand side of (6) is non positive, see [7]. Hence, *the closed loop is dissipative.* Our main result is that *the system is not dissipative if the exact computation of the fractional derivative is replaced by an approximate computation,* see Section 4, due to the small time delay so introduced.

The proofs in this note are quite straightforward and do not use any advanced technique. The sole property we borrow from [5, 6] is that the solutions of the closed loop of (1) and (3) depend continuously on the initial data on finite time intervals, and are smooth if the data are smooth enough.

Our goal is to prove negative results. Hence it is sufficient to study the problem with

$$\Omega = (0,1)\,, \qquad \Gamma_r = \{1\}\,, \quad \Gamma_a = \{0\}\,.$$

Either $\mathcal{L}$ or $\hat{}$ are used to denote Laplace transforms. Moreover, in the context of the Laplace transform, we meet the powers λ^γ, $0 \le \gamma < 1$ and $\Re e\,\lambda > 0$. The symbol λ^γ always denotes the principal determination of the power, so that $|\arg \lambda^\gamma| \le \gamma\pi/2 < \pi/2$.

2 A complement to the lack of exponential stability

Without relying on any specific state space realization, we prove that an estimate

$$||(w(t;\cdot), w_t(t;\cdot))||_{\mathcal{X}_0} < \; Me^{-\gamma t}||(w(0;\cdot), w_t(0;\cdot))||_{\mathcal{X}_0} \tag{7}$$

with $\gamma > 0$ cannot hold.

We consider the special initial condition

$$w(0;\cdot) = 0\,, \qquad w_t(0;\cdot) = 1 \tag{8}$$

so that if (7) has to hold then we must have in particular

$$\max_{0\le x\le 1} |w(t;x)| \le ||w(t;\cdot)||_{H_1(0,1)} < \; Me^{-\gamma t}$$

thanks to the continuous immersion of $C^0(0,1)$ in $H^1(0,1)$. Hence we should have

$$|w(t;0)| < \; Me^{-\gamma t}||(w(0;\cdot), w_t(0;\cdot))||_{\mathcal{X}_0}\,.$$

If this held then the Laplace transform $\hat{w}(\lambda;0)$ would be bounded on a halfplane $\Re e\,\lambda > -\gamma/2$ where $\gamma > 0$. We show that this is not the case.

We recall (see [2]) that

$$\mathcal{L}(D_*^\alpha v) = \lambda^\alpha \hat{v}(\lambda) - v(0)/\lambda^{1-\alpha}$$

and we note, from (3) and (8), that

$$v(0) = w(0,0) = 0\,.$$

The following equality holds:

$$\hat{w}(\lambda;0) = 2\frac{1-\cosh\lambda}{\lambda}\cdot\frac{1}{e^\lambda(\lambda+\lambda^\alpha)+e^{-\lambda}(\lambda-\lambda^\alpha)} \tag{9}$$

Remark 1. At a formal level, the derivation of the previous formula is straightforward. It can be justified as usual: the computations are correct for a sequence (ϕ_n, ψ_n) of sufficiently smooth initial conditions, $(\phi_n, \psi_n) \to (0,1)$ in $\mathcal{X}_0$ (for example, we can take $\phi_n = 0$ and ψ_n of class C^∞ with compact support). We get a Laplace transform $\hat{w}_n(\lambda, 0)$ which converges to the function $\hat{w}(\lambda, 0)$ above. ∎

Due to the fact that $(1-\cosh\lambda) = O(\lambda^2)$, this function is bounded for $\lambda \to 0$ from $\Re e\,\lambda > 0$. Moreover, we prove that $\hat{w}(\lambda;0)$ does not have poles in $\Re e\,\lambda \ge 0$, $\lambda \ne 0$. We state this as a lemma for later use.

Lemma 2. *If* $\Re e\,\lambda \ge 0$, $\lambda \ne 0$, *then* $e^\lambda(\lambda+\lambda^\alpha)+e^{-\lambda}(\lambda-\lambda^\alpha) \ne 0$.

Proof. Let us consider a number $\lambda_0 = \rho e^{i\theta} = \rho[\cos\theta + i\sin\theta]$ in the closed right half plane. We have

$$|\lambda_0 + \lambda_0^\alpha|^2 = |\rho(\cos\theta + i\sin\theta) + \rho^\alpha(\cos\alpha\theta + i\sin\alpha\theta)|^2 = \\ \rho^2 + \rho^{2\alpha} + 2\rho^{1+\alpha}\cos(1-\alpha)\theta$$

while

$$|\lambda_0 - \lambda_0^\alpha|^2 = \rho^2 + \rho^{2\alpha} - 2\rho^{1+\alpha}\cos(1-\alpha)\theta\,.$$

We recall that $\alpha \in (0,1)$ and $\theta \in [0,\pi/2]$. It follows that

$$\left|e^{-2\lambda_0}\right| \cdot |\lambda_0 - \lambda_0^\alpha|^2 \leq |\lambda_0 - \lambda_0^\alpha|^2 < |\lambda_0 + \lambda_0^\alpha|^2 \leq \left|e^{2\lambda_0}\right| \cdot |\lambda_0 + \lambda_0^\alpha|^2\,.$$

This shows that λ_0 cannot be a zero of the denominator. ∎

In spite of this, we use Rouché theorem in order to prove:

Proposition 3. *There exists a sequence* $\{\lambda_k\}$ *such that* $\Re e\, \lambda_k \to 0$ *and such that*

$$|\hat{w}(\lambda_k;0)| \to +\infty\,.$$

Proof. We write the denominator of the second factor in (9) as

$$\lambda e^{-\lambda}\left[e^{2\lambda} + 1 + \frac{1}{\lambda^{1-\alpha}}\left(e^{2\lambda} - 1\right)\right]$$

and we consider

$$f(\lambda) = e^{2\lambda} + 1\,, \qquad h(\lambda) = \frac{1}{\lambda^{1-\alpha}} g(\lambda)\,, \qquad g(\lambda) = \left(e^{2\lambda} - 1\right)\,.$$

Let

$$\lambda \in C_k\,, \qquad C_k = (k\pi + \pi/2)i + \rho e^{i\theta}\,, \qquad 0 \leq \theta \leq 2\pi\,.$$

Let D_k be the open disk bounded by C_k. Clearly,

$$m_k = \min_{C_k} |f(\lambda)| = m(\rho) \qquad \text{does not depend on } k$$

$$M_k(\rho) = \max_{C_k} |g(\lambda)| \leq e^{2\rho} + 1 = M(\rho) \qquad \text{does not depend on } k.$$

Consequently, for k large and $\rho < 1$ kept fixed,

$$\max_{C_k} |h(\lambda)| < \frac{M(\rho)}{(k\pi + \pi/2) - \rho}$$

and, for every ρ, we can find k so large that

$$\max_{C_k} |h(\lambda)| < m(\rho) \leq \min_{C_k} |f(\lambda)|\,.$$

This shows the existence of a sequence of zeros z_k of the denominator, which approaches the imaginary axis, $z_k \in D_k$.

We now prove that the numerator does not approach zero in D_k provided that the radius $\rho < \rho_0$, ρ_0 independent of k.

We fix a number ρ and we consider any $r \in [0, \rho)$. Let

$$\lambda = (k\pi + \pi/2)i + re^{i\theta}$$

be a point of D_k. Clearly, on D_k we have

$$\cosh\lambda = \begin{cases} \frac{1}{2}\left[-ie^{re^{i\theta}} + ie^{-re^{i\theta}}\right] & \text{if } k \text{ is odd} \\ \frac{1}{2}\left[ie^{re^{i\theta}} + ie^{-re^{i\theta}}\right] & \text{if } k \text{ is even}. \end{cases}$$

Note that dependence on k disappeared, and that we have two cases:

$$\lim_{\rho\to 0} |1 - \cosh\lambda| = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 1 - i & \text{if } k \text{ is even,} \end{cases}$$

uniformly with respect to θ. Existence of ρ_0 follows from here.

This shows that the Laplace transform of $\hat{w}(\lambda; 0)$ is unbounded in the halfplane $\Re e\,\lambda > -\sigma$, for every $\sigma > 0$, as wanted. ∎

3 Lack of continuous dependence when $T = +\infty$

In this section we consider the system

$$\begin{array}{ll} w_{tt} = w_{xx} \qquad 0 < x < 1\,, \qquad w(t,1) = 0 \\ w_x(t,0) = D_*^\alpha w(t,0) + g(t) \end{array} \tag{10}$$

with null initial condition

$$w(0,x) = 0\,, \qquad w_t(0,x) = 0\,.$$

We are going to prove that the linear transformation $g \to (w, w_t)$ *is not* continuous from $L^2(0,+\infty)$ to $L^2(0,+\infty; \mathcal{X}_0)$. In fact we prove that the transformation from g to the first component $w(t,\cdot) \in H^1(0,1)$ is not continuous.

We know from [5, 6] that the solution w exists but we are not really interested in this. If the solution does not exist then we are done (see also the following Remark 4). We formally compute the Laplace transform of $w(t;x)$. It is easily seen that

$$\hat{w}(\lambda; x) = \frac{e^{\lambda x} - e^{2\lambda}e^{-\lambda x}}{\lambda - \lambda^\alpha + e^{2\lambda}(\lambda + \lambda^\alpha)}\hat{g}(\lambda)\,.$$

We know from Lemma 2 that the denominator is non zero in $\Re e\,\lambda \geq 0$. We compute the derivative with respect to x and we see that

$$\begin{aligned} \hat{w}_x(\lambda; x) &= \frac{\lambda e^{\lambda x} + \lambda e^{2\lambda}e^{-\lambda x}}{\lambda - \lambda^\alpha + e^{2\lambda}(\lambda + \lambda^\alpha)}\hat{g}(\lambda) \\ &= \frac{1}{e^{-\lambda}\left[1 - \frac{1}{\lambda^{1-\alpha}}\right] + e^{\lambda}\left[1 + \frac{1}{\lambda^{1-\alpha}}\right]}\left[e^{-\lambda}e^{\lambda x} + e^{\lambda}e^{-\lambda x}\right]\hat{g}(\lambda)\,. \end{aligned} \tag{11}$$

We first compute the $L^2(0,1)$–norm of the bracket:

$$\begin{aligned}
&\int_0^1 \left| e^{-\lambda} e^{\lambda x} + e^{\lambda} e^{-\lambda x} \right|^2 \,\mathrm{d}x \\
&= \int_0^1 \left[e^{-\lambda} e^{\lambda x} + e^{\lambda} e^{-\lambda x} \right] \cdot \left[e^{-\bar\lambda} e^{\bar\lambda x} + e^{\bar\lambda} e^{-\bar\lambda x} \right] \,\mathrm{d}x \\
&= \int_0^1 \left\{ e^{-\lambda-\bar\lambda} e^{(\lambda+\bar\lambda)x} + e^{-\lambda+\bar\lambda} e^{(\lambda-\bar\lambda)x} + e^{\lambda-\bar\lambda} e^{(\bar\lambda-\lambda)x} + e^{\lambda+\bar\lambda} e^{-(\lambda+\bar\lambda)x} \right\} \,\mathrm{d}x \\
&= e^{-\lambda-\bar\lambda} \frac{1}{\lambda+\bar\lambda} \left[e^{\lambda+\bar\lambda} - 1 \right] + e^{-\lambda+\bar\lambda} \frac{1}{\lambda-\bar\lambda} \left[e^{\lambda-\bar\lambda} - 1 \right] \\
&+ e^{\lambda-\bar\lambda} \frac{1}{\bar\lambda-\lambda} \left[e^{\bar\lambda-\lambda} - 1 \right] + e^{\lambda+\bar\lambda} \frac{1}{\lambda+\bar\lambda} \left[1 - e^{-(\lambda+\bar\lambda)} \right] . \qquad (12)
\end{aligned}$$

We now choose $\lambda = \lambda_n + \epsilon_n$, $\lambda_n = (2n\pi + \pi/2)i$, $\epsilon_n = 1/n^2$. Note that $e^{\lambda_n} = i$, $e^{-\lambda_n} = -i$. We first consider the constant factor in (11), and replace $\lambda = \lambda_n + 1/n^2$. The denominator is

$$\left(-ie^{-1/n^2} + ie^{1/n^2} \right) + \frac{1}{(\lambda_n + 1/n^2)^{1-\alpha}} \left[ie^{1/n^2} + ie^{-1/n^2} \right] \longrightarrow 0 \qquad \text{for } n \to +\infty$$

of order $1/n^{1-\alpha}$.

We now replace $\lambda = \lambda_n + (1/n^2)$ in (12) (of which we have to take the square root). We note that

$$\lambda_n - \bar\lambda_n = (4n+1)\pi i \,, \qquad (\lambda_n + \epsilon_n) + \overline{(\lambda_n + \epsilon_n)} = 2\epsilon_n = 2/n^2 \,.$$

Hence, the second and third terms tend to 0 and we remain with

$$\begin{aligned}
&e^{-\lambda-\bar\lambda} \frac{1}{\lambda+\bar\lambda} \left[e^{\lambda+\bar\lambda} - 1 \right] + e^{\lambda+\bar\lambda} \frac{1}{\lambda+\bar\lambda} \left[1 - e^{-(\lambda+\bar\lambda)} \right] \\
&= \frac{e^{-2\epsilon_n}}{2\epsilon_n} \left[e^{2\epsilon_n} - 1 \right] + \frac{e^{2\epsilon_n}}{2\epsilon_n} \left[1 - e^{-2\epsilon_n} \right] \longrightarrow 2 \,.
\end{aligned}$$

This proves that

$$\limsup_k ||\hat w_x(\lambda_k;\cdot)||_{L^2(0,1)} = +\infty \,,$$

as wanted.

Remark 4. As we noted, we didn't prove existence of solutions of (10) and this proof is not needed for the negative result we are looking for. However, once formula (11) has been formally derived, we see that when $g(t)$ is square integrable both (11) and its product with λ are Laplace transforms (not of a square integrable function, of course) so that we can take formula (11) as the definition of the solution. ∎

4 Lack of robustness of dissipativity under numerical computation of the derivatives

As we noted, system (1) with u given by (3), is dissipative. But (3), as (2), requires the computation of the derivative w_t; hence the introduction of time delays in the case the derivative is computed numerically. So, it is natural to investigate whether dissipativity is retained under numerical computation of the derivative. It is known that dissipativity is affected by the introduction of delays but here the incremental quotient has to be introduced under a fractional integral so that its effect seems not having been studied. We prove in this section that even in this case dissipativity is not retained.

We shall now study the system

$$\left\{\begin{array}{ll} w_{tt}(t,x)=w_{xx}(t,x) & \text{in } [0,\,T]\times[0,\,1] \\ w(t,1)\equiv 0 & \\ w_x(t,0)=u(t) & \\ w(0,x)=a_1(x) & \\ w_t(0,x)=a_2(x)\,, & \end{array}\right. \tag{13}$$

where

$$u(t)=u(t;v(\cdot))=J^{\alpha}\left(\mathcal{Q}v\right) \tag{14}$$

and

$$v(t)=\left\{\begin{array}{ll} w(t,0) & \text{for } t\geq 0 \\ 0 & \text{for } t<0\,, \end{array}\right. \qquad \left(\mathcal{Q}v\right)(t)=\frac{v(t)-v(t-h)}{h}\,.$$

The feedback (14) is the counterpart of (3) where the fractional derivative operator D_*^{α} has been substituted by a numerical approximation.

We recall the expression (6) for the energy dissipated by the system (13) at a certain time T. Now the derivative in the feedback loop has been replaced with the incremental quotient and

$$-E(T)+E(0)=-\int_0^T u(t;v(\cdot))\,v'(t)\,\mathrm{d}t\,.$$

Our purpose is to show the existence of a time T and a function $\tilde{v}$ such that (6) is negative, i.e. the system gains energy. For this result to be useful, however, we must also prove that there exists a couple of initial conditions (a_1,a_2) which produces a "smooth" solution of the closed loop and such that the solution of (13) satisfies

$$w(t,0)=\tilde{v}\,. \tag{15}$$

To solve the latter problem, let us consider this auxiliary system (we will drop the tilde for simplicity):

$$\begin{cases} W_{\tau\tau}(\tau,\xi) = W_{\xi\xi}(\tau,\xi) \quad \text{in } [0,\,1]\times[0,\,\Xi] \\ W(0,\xi) = v(\xi) \\ W_\tau(0,\xi) = u(\xi) = u(\xi; v(\cdot)) \\ W(\tau,\Xi) = 0 \\ W(1,\xi) = 0\,. \end{cases} \tag{16}$$

We choose

$$a_1(\tau) = W(\tau,0) \qquad \text{and} \qquad a_2(\tau) = W_\xi(\tau,0)\,. \tag{17}$$

If it happens that a_1 and a_2 belong to H^1 and L^2 respectively, then the function

$$w(t,x) = W(x,t)$$

is the solution of system (13) and meets condition (15) with $\Xi = T$.

The functions u and v are initial conditions and can be chosen at will. So we can impose $u(\xi) = u(\xi, v(\cdot))$. We interpret $a_1(\tau)$ as a "control" so that the problem (16) is a kind of null controllability in time $\tau = 1$, for a suitably chosen value of Ξ. It is known that this problem is solvable, but $a_2(\tau)$ cannot be imposed at will and the crux of the matter is the proof that, for a suitably chosen functions v, the trace $a_2(\tau) = W_\xi(\tau,0)$ is well defined as an L^2 function. This is the statement of the following Theorem:

Theorem 1. *Let* $\Xi = 1$. *Let* v *have the following properties:*

1. *it is continuous on* $[0,1]$ *and piecewise differentiable, with piecewise continuous derivatives;*
2. $v(0) = 0$*;*
3. $v(T) = 0$*;*

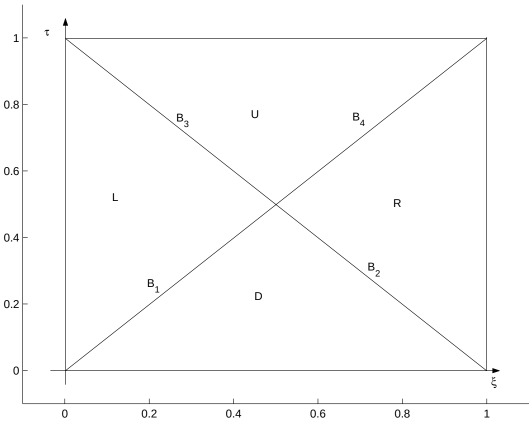
It is possible to find a control $a_1(\tau) \in H^1(0,1)$ *which drives* $W(\tau,\cdot)$ *to zero in time* 1 *and such that* $a_1(\tau)$*, i.e. the trace of* $W_\xi(\tau,\xi)$ *on* $\xi = 0$*, is piecewise continuous, hence square integrable.*

Proof. To prove the theorem we shall explicitly compute W. In order to do that, let us subdivide the domain into four triangles D, L, R and U according to the following figure 1.

We note that on each triangle, for a well known property of the wave equation, the solution can be written in the form

$$W(\tau,\xi) = \Phi(\xi+\tau) + \Psi(\xi-\tau)\,. \tag{18}$$

Figure 1:



The solution W in the subdomain D is given by D'Alambert formula:

$$W(\tau,\xi)=\frac{1}{2}[v(\xi+\tau)+v(\xi-\tau)]+\frac{1}{2}[F(\xi+\tau)-F(\xi-\tau)]\,,$$

where F a primitive of u. We choose the one such that $F(0)=0$. We can now calculate the trace of this solution on the two segments B_1 and B_2:

$$\begin{aligned}
R(\tau)&=W(\tau,\tau)=\frac{1}{2}[v(2\tau)+v(0)]+\frac{1}{2}[F(2\tau)-F(0)]\\
&=\frac{1}{2}[v(2\tau)+F(2\tau)] \qquad (19)\\
S(\tau)&=\frac{1}{2}[v(1-2\tau)+F(1)-F(1-2\tau)] \qquad (20)
\end{aligned}$$

for $\tau\in(0,1/2)$. Note that $R(1/2)=S(1/2)$, thanks to the conditions $v(0)=v(1)=0$.

Now we will consider the triangle L. Using expression (18) we impose continuity with the solution in D and $W(\tau,0)=a_1(\tau)$ (the function $a_1(\tau)$ is still to be determined). We get

$$W(\tau,\xi)=R\left(\frac{\tau+\xi}{2}\right)-R\left(\frac{\tau-\xi}{2}\right)+a_1(\tau-\xi)\,.$$

The trace of this expression on B_3 is

$$W(\tau,1-\tau)=R\left(\frac{1}{2}\right)-R\left(\tau-\frac{1}{2}\right)+a_1(2\tau-1)\quad\text{for }\tau\in(1/2,1)\,. \qquad (21)$$

Analogous computations made on R yield

$$W(\tau,\xi)=S\left(\frac{\tau-\xi+1}{2}\right)-S\left(\frac{\tau+\xi-1}{2}\right)$$

and

$$W(\tau,\tau)=S\left(\frac{1}{2}\right)-S\left(\tau-\frac{1}{2}\right)\quad \text{on } B_4\,. \tag{22}$$

Note that the condition $v(0)=0$ implies that the still unspecified function $a_1(\tau)$ must satisfy $a_1(0)=0$ and this implies equality at the common point of the restrictions of $W(\tau,\xi)$ to B_3 and, respectively, B_4.

To calculate the solution on U, we will use formula (18) again, completing it with the continuity conditions (21) and (22). We get

$$W(\tau,\xi)=R\left(\frac{1}{2}\right)-R\left(\frac{\tau-\xi}{2}\right)-S\left(\frac{\tau+\xi-1}{2}\right)+a_1(\tau-\xi)\,. \tag{23}$$

We combine (23) with (19) and (20). We see that $W(1,\xi)$ is given by

$$W(1,\xi)=R\left(\frac{1}{2}\right)-S\left(\frac{\xi}{2}\right)-R\left(\frac{1-\xi}{2}\right)+a_1(1-\xi)=$$

$$=-v(1-\xi)+a_1(1-\xi)\,.$$

We recall that we want $W(1,\xi)=0$. We must choose for this

$$a_1(\tau)=v(\tau)$$

so that $a_1(\tau)$ has the same regularity properties as $v(\tau)$.

We still have to see whether the condition $W_\xi(\tau,0)\in L^2$ is satisfied. A direct computation gives that $W_\xi(\tau,\xi)$ is equal to

$$\frac{1}{2}[v'(\xi+\tau)+v'(\xi-\tau)+u(\xi+\tau)-u(\xi-\tau)]\qquad \text{in } D;$$

$$\frac{1}{2}R'\left(\frac{\tau+\xi}{2}\right)+\frac{1}{2}R'\left(\frac{\tau-\xi}{2}\right)-v'(\tau-\xi)$$

$$=\frac{1}{2}[v'(\tau+\xi)-v'(\tau-\xi)+u(\tau+\xi)+u(\tau-\xi)]\qquad \text{in } L;$$

$$\frac{1}{2}R'\left(\frac{\tau-\xi}{2}\right)-\frac{1}{2}S'\left(\frac{\tau+\xi-1}{2}\right)-v'(\tau-\xi)$$

$$=\frac{1}{2}[v'(2-\tau-\xi)-v'(\tau-\xi)-u(2-\xi-\tau)+u(\tau-\xi)]\quad \text{in } U\,.$$

Our assumption is that $v(\tau)$ is piecewise differentiable with piecewise continuous derivative so that $u(\tau)$ is continuous, thanks to the smoothing properties of the fractional integral, so we can simply compute the trace of W_ξ on the left side of the domain as restriction. We get

$$W_\xi(\tau,0)=\begin{cases} v'(0) & \text{for } \tau=0\\ u(\tau) & \text{for } 0<\tau<1\\ 0 & \text{for } \tau=1\end{cases}$$

We can easily see that $a_2(\tau)=W_\xi(\tau,0)$ is piecewise continuous, therefore it belongs to L^2. ∎

We go back to formula (6). We must find a function v which meets the conditions of this theorem and such that, for some value of $h>0$,

$$-E(T)+E(0)=\int_0^1 \left[J^\alpha \mathcal{Q}v\right](t)v'(t)\,\mathrm{d}t<0\,,\qquad \mathcal{Q}v=\frac{v(t)-v(t-h)}{h}\,. \tag{24}$$

Of course, $v(t)$ will depend on h.

We choose $\alpha=1/2$ and we consider the following function:

$$v(t)=\begin{cases} 2t & \text{for } t\le h/2\\ 2h-2t & \text{for } h/2<t<h\\ 0 & \text{for } t\ge h\end{cases}\quad \text{hence}\quad v'(t)=\begin{cases} 2 & \text{for } t<h/2\\ -2 & \text{for } h/2<t<h\\ 0 & \text{for } t>h\,.\end{cases}$$

In this case, the integral in (24) is easily computed because $v'(t)=0$ for $t>h$ so that the argument of $v(t-h)$ is negative, i.e. $v(t-h)$ has to be replaced with zero. Integration by parts in the fractional integral shows that the integral in (24) is equal to (we ignore inessential multiplicative factors)

$$\begin{aligned}\frac{1}{h}\int_0^h v'(t)\left[\int_0^t v'(s)\sqrt{t-s}\,\mathrm{d}s\right]\,\mathrm{d}t &= \frac{1}{h}\int_0^{h/2} v'(t)\left[\int_0^t v'(s)\sqrt{t-s}\,\mathrm{d}s\right]\,\mathrm{d}t\\ &+\frac{1}{h}\int_{h/2}^h v'(t)\left[\int_0^t v'(s)\sqrt{t-s}\,\mathrm{d}s\right]\,\mathrm{d}t \end{aligned}\tag{25}$$

and we recall that $v'(t)$ is piecewise constant. Using this observation, both the integrals on the right hand side are easily computed (the last one as a sum of two) and we see that (25) is equal to

$$\frac{16}{15h}\left[4\left(\frac{h}{2}\right)^{5/2}-h^{5/2}\right]\,.$$

This is negative for every choice of $h>0$ so that our goal has been achieved.

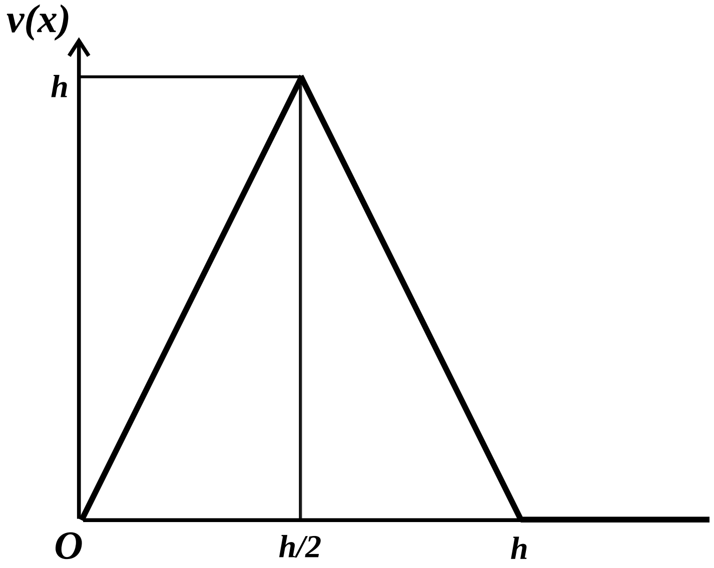


Figure 2: The graph of v

5 Appendix: Convergence of $J^\alpha f$ to f when $\alpha \to 0^+$

For completeness, in this appendix we prove the following result:

Theorem 5. *Let* $f(x)$ *be continuous on* $[0,T]$. *Then we have*

$$\lim_{\alpha\to 0^+} (J^\alpha f)(x) = f(x) \qquad (26)$$

uniformly on $[0,T]$.

Let instead $f(x)$ *be integrable on* $[0,T]$ *and continuous on* $(a,b) \subseteq [0,T]$. *Let* $[c,d]$ *be any closed interval contained in* (a,b). *Then equality (26) holds on* (a,b) *and the limit is uniform on* $[c,d]$

Proof. We examine first the case that $f(x)$ is continuous on $[0,A]$. We will make use of this formula from [2]:

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha} \qquad \forall\gamma > -1, \quad \forall\alpha > 0\,. \qquad (27)$$

This means that the result we want to prove holds for any polynomial. Moreover, if we let $\gamma = 0$ we find that $J^\alpha 1 \leqslant 2A$ if α is small enough, therefore

$$||J^\alpha f||_\infty \leqslant ||f||_\infty ||J^\alpha 1||_\infty \leqslant 2A||f||_\infty \qquad \forall f\,.$$

This means that J^α is continuous in uniform norm.

Let now f be an arbitrary continuous function and p a polynomial. Because of the previous result, we get

$$||f - J^\alpha f||_\infty \leqslant ||f - p||_\infty + ||p - J^\alpha p||_\infty + ||J^\alpha p - J^\alpha f||_\infty$$

$$\leq (2A + 1)||f - p||_\infty + ||p - J^\alpha p||_\infty \,.$$

Finally, thanks to Weierstrass approximation theorem, we may choose p such that $(2A+1)||f-p||_\infty < \epsilon/2$. Moreover, since we already know that the result holds for any polynomial, we may take $\alpha = \alpha(p)$ such that $||p-J^\alpha p||_\infty < \epsilon/2$.

Uniform convergence on $[0, T]$ is therefore proved.

We now consider the case that $f(x)$ is integrable on $[0, T]$ and continuous on (a, b). Since the result we want to prove concerns $\alpha \to 0^+$, in the following we will assume $\alpha < 1$. It is easily seen that we can find a sequence $\{f_n\} \in L^1(0, T) \bigcap C(0, T)$ such that:

- f_n converges to f in $L^1(0, T)$ norm;
- $f_n(x) = f(x)$ on (a, b).

Let us define $g_n(t) = f(t) - f_n(t)$. The following formula holds for $x \in [c, d]$:

$$J^\alpha g_n(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} g_n(t)\, \mathrm{d}t = \frac{1}{\Gamma(\alpha)} \int_0^a (x - t)^{\alpha-1} g_n(t)\, \mathrm{d}t$$

We note that $x - t > c - t > c - a$ so that on $[c, d]$ we have

$$|J^\alpha g_n(x)| \leq \frac{(c - a)^{\alpha-1}}{\Gamma(\alpha)} ||g_n||_{L^1(0,A)} \leq A||g_n||_{L^1(0,A)} \tag{28}$$

and the constant A does not depend on α (recall also $\lim_{\alpha\to0^+} \Gamma(\alpha) = 1$).

Now we consider that for $x \in [c, d]$ we have

$$|f(x) - J^\alpha f(x))| \leq |f_n(x) - f(x)| + |f_n(x) - J^\alpha f_n(x)| + |J^\alpha g_n(x)| =$$

$$= |f_n(x) - J^\alpha f_n(x)| + |J^\alpha g_n(x)| \,.$$

We use (28) and $||g_n||_{L^1(0,T)} \to 0$. Given ϵ we fix $n = n(\epsilon)$, independent of α, so large that the last addendum is less then $\epsilon/2$. With this value of n now fixed, we apply the first part of the theorem to the *continuous* function f_n: we can choose $\alpha = \alpha(n(\epsilon))$ such that the first addendum on the right hand side is less then $\epsilon/2$. This completes the proof.

References

[1] D. Datko, J. Lagnese, M. P. Polis, “An example on the effect of time delays in boundary feedback stabilization of the wave equations”, *SIAM J. Control Optim.* vol. 24, 1986, pp. 152-156.

[2] R. Gorenflo and F. Mainardi, “Fractional calculus: integral and differential equations of fractional order”, in *Fractals and fractional calculus in continuum mechanics,* A. Carpinteri and F. Mainardi Ed.s, CISM Courses and Lectures, 378, Springer, Vienna, 1997, pp. 223-276.

[3] V. Komornik, “Exact controllability and stabilization. The multiplier method”, in *Research in Applied Mathematics.* Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.

[4] H. Logemann, R. Rebarber, G. Weiss, “Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop”, *SIAM J. Control Optim.* vol. 34, 1996, pp. 572-600.

[5] B. Mbodje and G. Montseny, “Boundary fractional derivative control of the wave equation”, *IEEE T. Automat. Contr.* vol. 40, 1995, pp. 378-382.

[6] B. Mbodje, “Analyse et controle de certains modeles viscoelastiques via una realisation non hereditaire des operateurs integrodifferentiels fractionnaires”, Ph.D. Thesis presented at Laboratoire d’Analyse et d’Architecture des Systèmes du CNRS, Rapport LAAS 95026 1995.

[7] J.A. Nohel and D.F. Shea, “Frequency domain methods for Volterra equations”, *J. Math. Anal. Appl.* vol. 22, 1976, pp. 278-304.

[8] L. Pandolfi, Controllability properties of perturbed distributed parameter systems. *Linear Algebra Appl.,* vol. 122/123/124, 1989, pp. 525-538.

[9] I. Podlubny, “Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications”, in *Mathematics in Science and Engineering*, Academic Press, San Diego, CA, 1999.