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**HEAT EQUATION WITH MEMORY:
LACK OF CONTROLLABILITY TO THE REST**

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Heat equation with memory: lack of controllability to the rest ^{*}

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Abstract

We prove that the one dimensional heat equation with memory cannot be steered to the rest for large classes of memory kernels and controls. The approach is based on the application of the theory of interpolation in Paley–Wiener spaces.

1 Introduction

Controllability for the heat equations with memory is a recent field and many different “controllability” results have been proved, for example in [3, 13, 17, 18, 20, 22]. These papers prove several kind of controllability results for classes of equations of the following form (here $t > 0$ and $x \in (0, \pi)$):

$$\theta_t = \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s) ds, \quad \theta(0) = \xi \quad (1)$$

and θ is furthermore acted upon by a control, either distributed or boundary control. “Distributed control” is the case that a further additive term $u(t, x)$ appears in Eq. (1), with $u = u(t, x) \in L^2((0, T) \times (0, \pi))$ for each $T > 0$. We consider also the case of “boundary control” (of Dirichlet type), i.e. the case $\theta(t, 0) = u(t)$, $\theta(t, \pi) = 0$ (a similar treatment is also possible in the case of

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Neumann control, see below). In this case we require $u = u(t) \in L^2(0, T)$ for each $T > 0$. Exact controllability has been studied in the references above (also in the case of equations on regions of \mathbb{R}^n), i.e. it is studied whether any final target can be hit, by using a suitable control. In particular, the target can be 0 so that we get a kind of null controllability which however need not be controllability to the rest, since even if the trajectory hits 0, the solution may leave 0 in the future. Using a term which was once popular among specialists of systems with delays, this is a kind of “relative controllability”, see [9]. In contrast with this, we say that the system is *controllable to the rest* when for every initial condition ξ we can find a control u with compact support whose corresponding trajectory $\theta(t)$ has compact support too.

Controllability to the rest is not impossible, see Section 3, but we prove that it is an exceptional property in the presence of memory. This, may be, will not come as a surprise but our point in this paper is that the obstacle to the controllability to the rest is not so much the infinite memory of the equation but the joint facts that the equation has infinite memory and that $\theta(t)$ takes values in an infinite dimensional Hilbert space. We shall see in fact that if $\theta(t)$ takes values in \mathbb{R}^n and the term θ_{xx} is replaced by $A\theta$ (A is a matrix) then controllability is well possible also if the kernel $N(t)$ has the properties required by the negative results presented in Section 4.

Our results cover in particular the cases that the kernel $N(t)$ is a linear combination of exponentials or it is a kernel of Abel type, as studied for example in [7, 8]. The proof of the negative results is based on Fourier method and Laplace transform so to reduce controllability to a moment/interpolation problem in a Paley-Wiener space, see [1, 10]. This is done in Section 2 where all the relevant definitions are given. Preliminary examples are in Section 3: two finite dimensional examples show that controllability to the rest is possible in this case, in fact it is even a generic property. A third example shows a heat equation with memory, which is actually the wave equation “in disguise”, which is controllable to the rest. A fourth example shows in a concrete case the idea of this paper.

The negative results are in Section 4.

2 Statements of control problems and interpolation

Let us consider three control problems connected with (1).

Problem i) Boundary control.

$$\begin{aligned}\theta_t &= \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s) ds, \quad x \in (0, \pi), \quad t > 0, \\ \theta(t, 0) &= u(t), \quad \theta(t, \pi) = 0, \quad \theta(0, x) = \xi(x) \in L^2(0, \pi).\end{aligned}\tag{2}$$

The control u is square integrable on any interval $[0, T]$, $u \in L^2_{\text{loc}}[0, \infty)$.

Negative results as those in Section 4 can be obtained also in the case of Neumann boundary controls, with similar proofs.

Problem ii) Distributed control with a given profile.

$$\begin{aligned}\theta_t &= \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s) ds + b(x)u(t), \quad x \in (0, \pi), \quad t > 0 \\ \theta(t, 0) &= \theta(t, \pi) = 0, \quad \theta(0, x) = \xi(x) \in L^2(0, \pi).\end{aligned}\tag{3}$$

Here $b \in L^2(0, \pi)$ is a given function and u is a control.

Problem iii) Control distributed on a subdomain.

$$\begin{aligned}\theta_t &= \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s) ds + u(x, t), \quad x \in (0, \pi), \quad t > 0 \\ \theta(t, 0) &= \theta(t, \pi) = 0, \quad \theta(0, x) = \xi(x) \in L^2(0, \pi).\end{aligned}\tag{4}$$

Here u is a control supported (in x) on $[\beta, \gamma] \subset (0, \pi)$.

Now we give the following definition which apply to each one of the cases above. Solutions will be rigorously defined later on.

Definition 1 *The initial vector $\xi \in L^2(0, \pi)$ is controllable to the rest if we can find $T > 0$, a control $u(t)$ and a corresponding solution $\theta(t)$, with $u(t) = 0$ and $\theta(t) = 0$ for $t > T$.*

The system is controllable to the rest when every initial vector $\xi \in L^2(0, \pi)$ is controllable to the rest; if the controllability time T can be chosen independent of ξ then we say that the system is controllable to the rest in time T .

As we said we are going to prove that controllability to the rest, for the classes of kernels we consider, is an exceptional property, i.e. it is possible solely for very special initial conditions. We note that in principle such negative results do not make any use of the known theorems on existence/uniqueness of solutions of the heat equation with memory (as stated for example in [4, 7, 8, 20]): we simply prove that if a solution has a certain initial condition ξ and compact support, with control which has compact support too, then ξ must satisfy stringent requirements. In fact, this general observation applies to the cases **i)** and **ii)** while the results on case **iii)** in Section 4.2 are derived from the negative results concerning boundary control, thanks to regularity properties of the solution $\theta(t, x)$.

2.1 Paley-Wiener classes and interpolation problems

We first recall the definition of the Paley-Wiener class \mathcal{PW}^T . This is the linear space of the entire functions $\hat{\phi}(\lambda)$ which are of exponential type equal to or less than T in the whole complex plane, and square integrable on the imaginary axis.

Let $\mathcal{PW}_+^T = \mathcal{PW}^T \cap \mathcal{H}^2$ (we denote \mathcal{H}^2 the Hardy space of the right half plane $\Re z > 0$, with inner product $\int_{-\infty}^{\infty} f(ix)\bar{g}(ix) dx$). This is the image under the Laplace transform \mathcal{L} of the subspace of functions f from $L^2(0, \infty)$ supported on $[0, T]$

We put $\mathcal{PW}_+ = \bigcup_{T>0} \mathcal{PW}_+^T$. This space is the linear space of the entire functions $\hat{\phi}(\lambda)$ which are of exponential type equal to or less than some T in the left half plane, of exponential type 0 in the right half plane, and whose restrictions to the imaginary axis are square integrable. An important example is $[1 - \exp(-\lambda T)]/\lambda$.

Let the function $\theta(t)$ take values in a Hilbert space X . We say that $\hat{\theta}(\lambda)$ belongs to the class \mathcal{PW}_+ when the scalar value function $\langle \hat{\theta}(\lambda), x \rangle$ belongs to the class \mathcal{PW}_+ for every $x \in X$.

So, in order to control to the rest, we search for input functions u with $\hat{u}(\lambda)$ in the class \mathcal{PW}_+ such that the Laplace transform of the corresponding solution $\theta(t)$ to Equation (1) is in the class \mathcal{PW}_+ too.

Inspired by [1], we reduce controllability to the rest to an interpolation/moment problem. We consider Problems **i)** and **ii)** here since Problem **iii)** will be studied with different methods. For definiteness, let us consider the case of boundary control, Problem **i)**. The arguments below are easily adapted to the case of Problems **ii)** and **iii)**.

We first explain in what sense a function $\theta = \theta(t, x)$ is a solution of Eq. (2). We note the following equality which holds for every function $\psi(x) \in H^2(0, \pi)$ such that $\psi(0) = \kappa$, $\psi(1) = 0$ and for every $\phi(x) \in H^2(0, \pi) \cap H_0^1(0, \pi)$ (below $\langle \phi, \psi \rangle$ is the inner product in $L^2(0, \pi)$):

$$\langle \psi_{xx}, \phi \rangle = \phi'(0)\kappa + \langle \psi(\cdot), \phi_{xx}(\cdot) \rangle = \phi'(0)\kappa + \langle \psi, A\phi \rangle. \quad (5)$$

Here A is the operator

$$\text{dom } A = H^2(0, \pi) \cap H_0^1(0, \pi) \subseteq L^2(0, \pi), \quad (A\phi)(x) = \phi_{xx}(x) \in L^2(0, \pi). \quad (6)$$

This formula replaces the boundary condition with an additive term, and suggests the following definition: the function $\theta(t) = \theta(t, x) \in$

$L^2_{\text{loc}}[0, +\infty; L^2(0, \pi))$ solves Problem **i**) when the following equality holds for every $\phi \in \text{dom } A$:

$$\begin{aligned} \frac{d}{dt} \langle \theta(t), \phi \rangle &= \alpha [\langle \theta(t), A\phi \rangle + \phi'(0)u(t)] + \int_0^t N(t-s) [\langle \theta(s), A\phi \rangle \\ &+ \phi'(0)u(s)] ds, \quad \theta(0) = \xi. \end{aligned} \quad (7)$$

The normalized eigenvectors of A are the functions $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ (the corresponding eigenvalue is $-n^2$) and this is a complete set in $L^2(0, \pi)$. Hence we can expand

$$\theta(t) = \theta(t, x) = \sum_{n=1}^{+\infty} \theta_n(t) \phi_n(x). \quad (8)$$

If in particular $\phi = \phi_n$ in (7) we see that the n -th component $\theta_n(t) = \langle \theta(t), \phi_n \rangle$ satisfies

$$\begin{aligned} \theta'_n(t) &= \alpha [-n^2 \theta_n(t) + \phi'_n(0)u(t)] \\ &+ \int_0^t N(t-s) [-n^2 \theta_n(s) + \phi'_n(0)u(s)] ds, \quad \theta_n(0) = \xi_n = \langle \xi, \phi_n \rangle. \end{aligned} \quad (9)$$

By the definition, if the initial condition ξ can be controlled to the rest, then we can find a control u with compact support such that each one of the functions $\theta_n(t)$ which solves (9) must have compact support too.

We note that $\phi'_n(0) = \sqrt{\frac{2}{\pi}} n$ and we introduce the notation

$$G(\lambda) = \alpha + \mathcal{L}(N) = \alpha + \hat{N}(\lambda),$$

where \mathcal{L} and $\hat{\cdot}$ denote Laplace transform. We compute the Laplace transform of $\theta_n(t)$ and we get

$$\text{Problem i)} \quad \hat{\theta}_n(\lambda) = \frac{1}{\lambda + n^2 G(\lambda)} \left(\xi_n + \frac{n\sqrt{2}G(\lambda)}{\sqrt{\pi}} \hat{u}(\lambda) \right), \quad (10)$$

$$\text{Problem ii)} \quad \hat{\theta}_n(\lambda) = \frac{1}{\lambda + n^2 G(\lambda)} (b_n \hat{u}(\lambda) + \xi_n), \quad b_n = \langle b, \phi_n \rangle. \quad (11)$$

Hence, if controllability to the rest is possible then for any given initial condition $\xi = \theta(0) \in L^2(0, \pi)$, we can find an input $\hat{u}(\lambda) \in \mathcal{PW}_+$ such that $\hat{\theta}_n(\lambda)$ belongs to \mathcal{PW}_+ . In particular, the function $\hat{\theta}(\lambda)$ should not have a singularity

at the roots of the denominator $\lambda + n^2G(\lambda)$ of the expressions in (10)-(11). Hence we must have

$$\text{Problem i)} \quad \hat{u}(\lambda) = \sqrt{\frac{\pi}{2}} \frac{n}{\lambda} \xi_n \quad \text{when } \lambda + n^2G(\lambda) = 0, \quad (12)$$

$$\text{Problem ii)} \quad \hat{u}(\lambda) = -\frac{1}{b_n} \xi_n \quad \text{when } \lambda + n^2G(\lambda) = 0. \quad (13)$$

(We may suppose $b_n \neq 0$ for every n since if $b_m = 0$ for some m we cannot control the m -th component $\theta_m(t)$).

The interpolation problems (12) and (13) are also moment problems with respect to an exponential family. In fact, the condition (13), can be written as

$$\int_0^T e^{-\lambda t} u(t) dt = -\frac{1}{b_n} \xi_n \quad \text{when } \lambda + n^2G(\lambda) = 0 \quad (14)$$

(it is possible that for a fixed n the equation $\lambda + n^2G(\lambda) = 0$ has more than one root. In this case condition (14) must be satisfied at each root).

The moment problem which corresponds to (12) is

$$\int_0^T e^{-\lambda t} u(t) dt = \sqrt{\frac{\pi}{2}} \frac{n}{\lambda} \xi_n. \quad (15)$$

This interpolation condition must hold at the sequence of the points λ which satisfy $\lambda + n^2G(\lambda) = 0$.

Our negative results on the controllability to the rest will be derived since, under appropriate conditions, we can see that there are obstructions to the solution of the interpolation/moment problem. We list two of such obstructions, which will be explicitly used. The first is trivial.

Obstruction 1: we see directly from the definition that if there is a sequence λ_n of zeros of $\lambda + n^2G(\lambda)$ which converges to μ , then the interpolation problems (12) and (13) are not generally solvable. To show this, we choose $\xi_{2k+1} = 0$ and we note that $\hat{u}(\lambda)$ is an entire function so that the conditions $u(\lambda_{2k+1}) = 0$ and $\lambda_{2k+1} \rightarrow \mu$ imply that $\hat{u}(\lambda)$ has to be identically zero and this forces every component ξ_{2k} to be zero too: neither interpolation condition (12) nor (13) are possible unless $\xi = 0$.

A more powerful obstruction to interpolation follows from the following completeness condition.

Obstruction 2: Let $\{\lambda_n\}$ be a sequence of complex numbers with the following property: there exists a number $\gamma > 1$ such that

$$\sum_{n=1}^{+\infty} \frac{1}{|\lambda_n|^\gamma} = +\infty. \quad (16)$$

In this case the sequence of the exponentials $\{e^{\lambda_n t}\}$ is complete on every interval. See [21, p. 105 “complement” to Remark 2] for an even more general formulation.

Note that the condition in **Obstruction 1** implies condition (16).

Remark 1 In the case λ_n are in a sector not containing the imaginary axis, we may use the following statement, which is equivalent to the well known Müntz theorem.

Let $\{\lambda_n\}$ be a sequence of complex numbers in an acute angle containing the positive real axis. Assume

$$\sum_{n=1}^{+\infty} \frac{1}{|\lambda_n|} = +\infty. \quad (17)$$

Then for any $T > 0$ the set of the exponentials $e^{-\lambda_n t}$ is complete in $C(0, T)$ hence also in $L^2(0, T)$ (see [23, p. 90] and note the exchange of real and imaginary axis, due to the fact that the exponentials in [23] have the form $e^{i\lambda_n t}$).

The fact that λ_n should be separated by the imaginary axis is crucial: the family of the function e^{int} , $n > 0$ (note, $n = 0$ is excluded) is not complete in $L^2(0, \pi)$, in spite of the fact that $\sum_{n=1}^{+\infty} 1/|in| = +\infty$. Obstruction 2 is an extension of the completeness condition (17) to sequences which may approach or stay on the imaginary axis. ■

The condition in **Obstruction 2** implies that our interpolation/moment problem cannot be solved if ξ has to be an arbitrary initial condition in $L^2(0, \pi)$. In fact, we shall use also a stronger consequence in Remark 5 and in Theorem 9. For this reason, we state:

Lemma 2 *Let condition (16) be fulfilled. Let r be nonnegative and let us consider the moment problems (14) or (15) with $\xi_n = \langle \xi, \phi_n \rangle$, and ξ being any element in $\text{dom}(-A)^r$. These moment problems cannot be solved.*

Proof. We fix an index k and we consider the equalities required by the moment problems for every index $n \neq k$. Condition (16) is satisfied by the numbers λ_n , $n \neq k$.

We consider the special initial condition $\xi = c\phi_k$, where c is constant. Clearly, $\xi \in \text{dom}(-A)^r$. The moment equalities for $n \neq k$ are

$$\int_0^T e^{-\lambda_n t} u(t) dt = 0, \quad n \neq k. \quad (18)$$

Since the exponential family $\{e^{-\lambda_n t}, n \neq k\}$ is complete, this implies $u = 0$. Then the value of $\int_0^T e^{-\lambda_k t} u(t) dt$ is zero too and cannot be assigned at will. The moment problems (14) or (15) can only be solved if $c = 0$. ■

3 Preliminary examples

We first consider two special examples of systems evolving in a finite dimensional spaces. We shall see that the kernels $N(t)$ in these examples are prototypes of the kernels of the negative results presented in Section 4. In spite of this, we shall see that controllability to the rest is now not only possible: controllability to the rest always holds in Example 2 and holds generically in the case of Example 1. In order to stress the difference between systems evolving in finite dimensional spaces and in infinite dimensional Hilbert spaces, we use x instead of θ in the first case.

We recall that the pair (A, B) of an $n \times n$ matrix A and of an $n \times m$ matrix B is called λ -controllable when Hautus condition holds at λ , i.e. when

$$\ker \begin{bmatrix} \lambda I - A^* \\ B^* \end{bmatrix} = \{0\}.$$

If λ -controllability holds at every λ then the pair (A, B) is called “controllable” and this is equivalent to the fact that the solution of the system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

can be steered to the rest for every initial condition x_0 .

We are interested in the first example since in this case $\hat{K}(\lambda)$ admits zeros.

Example 1 The first example is

$$\dot{x} = Ax + A \int_0^t [e^{\gamma(t-s)} + e^{\sigma(t-s)}] x(s) ds + Bu(t) \quad (19)$$

where $\gamma \neq \sigma$. We assume that B is not surjective and that the pair (A, B) is controllable. We introduce

$$y(t) = \int_0^t e^{\gamma(t-s)} x(s) ds, \quad z(t) = \int_0^t e^{\sigma(t-s)} x(s) ds$$

and we see that system (19) can be controlled to the rest provided that we can find $T > 0$ and a control u which is zero for $t > T$ and such that

$$x(T) = 0, \quad y(T) = 0, \quad z(T) = 0.$$

In fact let $u(t) = 0$ for $t = T + \tau$. Then $x(T + \tau)$ solves

$$\begin{aligned} \dot{x}(T + \tau) &= Ax(T + \tau) + A \int_0^\tau [e^{\gamma(\tau-r)} + e^{\sigma(\tau-r)}] x(T + r) dr \\ &+ A \{e^{\gamma\tau} y(T) + e^{\sigma\tau} z(T)\}. \end{aligned}$$

The solution $\tau \rightarrow x(T + \tau)$ is identically zero if it happens that $x(T) = 0$, $y(T) = 0$ and $z(T) = 0$ (and, in fact, solely in this case). Now we observe that a state space realization of the map from u to x is

$$x' = A(x + y + z) + Bu, \quad y' = \gamma y + x, \quad z' = \sigma z + x. \quad (20)$$

The observation is that this system is controllable provided that the pair (A, B) is controllable unless γ and σ are algebraically related: if system (20) is *not* controllable then there exists a particular value of λ such that we can find a nonzero solution of the system

$$\begin{bmatrix} \lambda I - A^* & -I & -I \\ -A^* & (\lambda - \gamma)I & 0 \\ -A^* & 0 & (\lambda - \sigma)I \\ B^* & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \quad (21)$$

We investigate 0-controllability first. We subtract the third line from the first and second ones and we see that system (21) with $\lambda = 0$ is equivalent to

$$\begin{bmatrix} 0 & -I & (\sigma - 1)I \\ 0 & -\gamma I & \sigma I \\ -A^* & 0 & -\sigma \\ B^* & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \quad (22)$$

In the special case $\sigma = 1$ then $y = 0$ and, from the second row, $z = 0$. The pair (A, B) being controllable, it follows also $x = 0$. Let instead σ be different from 1. We choose any $x \neq 0$ in the kernel of B^* and we define $z = -(1/\sigma)A^*x$. Note that $z \neq 0$ because the pair (A, B) is controllable. We see that controllability does not hold if and only if there exists a solution y to the system

$$y = (\sigma - 1)z, \quad \gamma y = \sigma z.$$

We have $\gamma \neq 0$ because $z \neq 0$ and we see that a non null solution to this system exists if and only if $\sigma - 1 = \sigma/\gamma$. If this relation holds then we don't have 0-controllability.

The properties of γ -controllability and σ -controllability always hold.

We consider now the case $\lambda \notin \{0, \gamma, \sigma\}$. We subtract the third line from the first and second one. We see that $y = [(\lambda - \sigma)/(\lambda - \gamma)]z$. The first line gives $x = \frac{1}{\lambda} \left\{ (\lambda + 1 - \sigma) + \frac{\lambda - \sigma}{\lambda - \gamma} \right\} z$. So, in order not to have λ -controllability we should have

$$-(\lambda - \sigma)z = A^* \frac{1}{\lambda} \left\{ (\lambda + 1 - \sigma) + \frac{\lambda - \sigma}{\lambda - \gamma} \right\} z, \quad (23)$$

$$B^* \frac{1}{\lambda} \left\{ (\lambda + 1 - \sigma) + \frac{\lambda - \sigma}{\lambda - \gamma} \right\} z = 0. \quad (24)$$

This is not possible with $z \neq 0$, because (A, B) is controllable, unless

$$\left\{ (\lambda + 1 - \sigma) + \frac{\lambda - \sigma}{\lambda - \gamma} \right\} z = 0;$$

but, in this case equality (23) gives $\lambda = \sigma$, while we explicitly assumed $\lambda \neq \sigma$.

We sum up: if the system is not controllable to the rest then γ and σ are related by

$$\gamma \neq 0, \quad \sigma \neq 0 \quad \sigma + \gamma = \sigma\gamma$$

(the condition $\gamma \neq 0$ is implied by the next ones and it is listed for the sake of symmetry). Note that we are *not* asserting that if the previous condition holds then the system is *not* controllable to the rest because the previous conditions correspond to the controllability of the “augmented” system (20), with *arbitrary* initial condition while when studying system (19) we must have $y(0) = 0$ and $z(0) = 0$. ■

In the second example we consider the case $\hat{K}(\lambda)$ without zeros but with a double pole.

Example 2 Here we consider x scalar and

$$N(t) = t \quad \text{i.e.} \quad \hat{k}(\lambda) = \frac{1}{\lambda^2}.$$

The system is

$$x' = \int_0^t (t-s)x(s) ds + u(t), \quad x(0) = x_0.$$

As in example (1), we see that controllability to the rest holds if we can find $T > 0$ and $u(t)$ such that both the following conditions hold:

$$\begin{cases} x(T) = 0, \\ y(T) = 0, \\ \zeta(T) = 0 \end{cases} \quad \text{where} \quad \begin{cases} y(t) = \int_0^t (t-s)x(s) ds \\ \zeta(t) = \int_0^t x(s) ds. \end{cases}$$

So we study the controllability of the system

$$\dot{x} = y + u, \quad \dot{y} = -\zeta, \quad \dot{\zeta} = x.$$

We omit the simple verification that this system is λ -controllable for every λ . ■

We consider now an example of a system of the form (1) which is controllable to the rest. This is an integrated form of the wave equation.

Example 3 We consider

$$\theta_t = \int_0^t \theta_{xx}(s) \, ds, \quad \theta(0) = \xi, \quad \theta(t, 0) = u(t), \quad \theta(t, 1) = 0.$$

We introduce

$$y(t) = \int_0^t \theta_{xx}(s) \, ds$$

so that we get the controllable system

$$\theta_t = y, \quad y_t = \theta_{xx}$$

i.e. the wave equation. ■

Our point in this paper is that every system of the form (1) which is controllable to the rest and which has a “rational kernel”, i.e. a kernel with rational Laplace transform, is a wave equation in disguise.

The idea that we use in this paper is contained in the last example:

Example 4 We consider now Problem **i**) with the kernel $N(t) = 1 + t$ and $\alpha = 0$, i.e.

$$\hat{N}(\lambda) = \frac{1}{\lambda} + \frac{1}{\lambda^2}.$$

The equation to be solved in order to identify the interpolation points is

$$n^2 \left\{ \frac{1}{\lambda} + \frac{1}{\lambda^2} \right\} = -\lambda \quad \text{i.e.} \quad n^2 \{\lambda + 1\} = -\lambda^3.$$

For every number n this equation has one solution in the interval $[-1, -1 + 1/n]$. In fact, let

$$h(\lambda) = n^2(\lambda + 1) + \lambda^3. \quad \text{We have} \quad \begin{cases} h(-1) = -1 \\ h(-1 + 1/n^2) = \frac{3}{n^2} - \frac{3}{n^4} + \frac{1}{n^6} > 0. \end{cases}$$

So, there is a chain of roots $\{\lambda_n^{(1)}\}$ which accumulates to -1 and interpolation at these roots is impossible, see **Obstruction 1**.

In fact, we have two more chains of roots:

$$\lambda_n^{(2)} = -in + \chi_1^{(n)}, \quad |\chi_1^{(n)}| < 1, \quad \lambda_n^{(3)} = in + \chi_2^{(n)}, \quad |\chi_2^{(n)}| < 2.$$

This is easily seen using Rouchè Theorem. So, controllability to the rest is equivalent to the following interpolation problem:

$$\begin{aligned}\hat{u}(\lambda_n^{(1)}) &= -\frac{\sqrt{\pi}}{n} \frac{[\lambda_n^{(1)}]^2}{1 + \lambda_n^{(1)}} \xi_n, \\ \hat{u}(\lambda_n^{(2)}) &= -\frac{\sqrt{\pi}}{n} \frac{[\lambda_n^{(2)}]^2}{1 + \lambda_n^{(2)}} \xi_n, & \hat{u}(\lambda_n^{(3)}) &= -\frac{\sqrt{\pi}}{n} \frac{[\lambda_n^{(3)}]^2}{1 + \lambda_n^{(3)}} \xi_n.\end{aligned}$$

The interpolation problem described at the second line is solvable. The obstruction to controllability is due solely to the chain $\{\lambda_n^{(1)}\}$.

4 Lack of controllability to the rest

We now consider the case that θ evolves in a separable Hilbert space H and we present the negative results. It is convenient to consider the case of boundary controls, and the case of a distributed control with a profile, i.e. the cases in which the control takes values in \mathbb{R} , first.

4.1 The Problems i) and ii)

Example 3 shows that system (1) can be controllable to the rest. However, our negative results show that this is an exceptional case. The first negative result is as follows:

Theorem 3 *If there exists a zero λ_0 of $G(\lambda) = \alpha + \hat{N}(\lambda)$ then controllability to the rest in the cases i), i.e. Eq. (2), and ii), Eq. (3), is impossible.*

Proof. We shall see the existence of a sequence $\{\lambda_n\}$ of zeros of

$$\lambda + n^2 G(\lambda),$$

one λ_n for each n , which is convergent. **Obstruction 1** then shows that interpolation is impossible.

The existence of the sequence $\{\lambda_n\}$ follows from Rouchè Theorem, as follows: Let λ_0 be a zero of $G(\lambda)$. We consider a disk centered at λ_0 in which $G(\lambda)$ has no singularity and such that on the boundary $|G(\lambda)| > \mu > 0$. Let $\nu = \max |\lambda|$ on the boundary of the disk. Clearly, for n large enough we have $n^2 \mu > \nu$ so that the function $n^2 G(\lambda) + \lambda$ has a zero λ_n in this disk, for every large n . So, as we noted, interpolation is impossible. ■

As an example, if $\alpha > 0$ and $k(t) = e^{\gamma t}$, or if $\alpha = 0$ and $k(t) = e^{\gamma t} + e^{\sigma t}$ then controllability to the rest is impossible. This is to be contrasted with Example 1.

An example of a function $G(\lambda)$ without zeros is $G(\lambda) = \hat{k}(\lambda) = 1/\lambda^2$ (hence $\alpha = 0$) as in Example 2 and this case is not covered by Theorem 3.

Theorem 4 *Let*

$$G(\lambda) = \frac{1}{\lambda^\nu + g(\lambda)}$$

where $\nu \geq 2$ and $|g(\lambda)| < M|\lambda|^{\nu-1}$. The estimate for $g(\lambda)$ is assumed in a sector \mathcal{S} with vertex at 0 and containing a ray $\arg z = \frac{2\pi k + \pi}{\nu+1}$.

Under this assumption, controllability to the rest in the cases **i**), i.e. Eq. (2), and **ii**), Eq. (3), is impossible.

Proof. The proof depends on **Obstruction 2**.

The equation $\lambda + n^2 G(\lambda) = 0$ gives

$$0 = (\lambda^{\nu+1} + n^2) + \lambda g(\lambda). \quad (25)$$

We compare $\lambda g(\lambda)$ and

$$f_n(\lambda) = \lambda^{\nu+1} + n^2.$$

The zeros of $f_n(\lambda)$ are $(-n^2)^{1/(\nu+1)}$ and lies on the lines identified by the $(\nu+1)$ -roots of (-1) , $e^{(i\pi+2ki\pi)/(\nu+1)}$, $0 \leq k \leq \nu$. By assumption, at least one of them belong to the sector \mathcal{S} . Let this root be $e^{i\pi(2k_0+1)/(\nu+1)}$.

Consider the sequence of the roots

$$\zeta_n = n^{2/(\nu+1)} e^{i\pi(2k_0+1)/(\nu+1)}.$$

These belong to a straight line in the sector \mathcal{S} . Moreover, there exists a number $\gamma > 1$ such that $\sum 1/|\zeta_n|^\gamma = +\infty$ since $2/(\nu+1) < 1$. **Obstruction 2** implies that for set of the zeros of the functions $f_n(\lambda)$ our interpolation problems cannot be solved generally.

We prove that this negative property is inherited by the zeros of $\lambda + n^2 G(\lambda)$ as follows: using again Rouché Theorem, we prove that there exists a number $\sigma > 0$ and a number N_0 such that for $n > N_0$ the function $\lambda + n^2 G(\lambda)$ has a zero μ_n in a disk of radius σ centered at each zero of ζ_n . So, we have also $\sum 1/|\mu_n|^\gamma = +\infty$ and interpolation is impossible.

Let Γ_n be the circle

$$\Gamma_n : \quad \lambda = \zeta_n + \sigma e^{i\omega}, \quad \omega \in [0, 2\pi].$$

The number σ is still to be determined. We compare $|\lambda|^\nu$ with $f_n(\lambda) = (\lambda^{\nu+1} + n^2)$ on this circle.

We obtain

$$\begin{aligned} |\lambda|^\nu &= |\zeta_n + \sigma e^{i\omega}|^\nu \leq n^{2\nu/(\nu+1)} \left[1 + \frac{\sigma}{n^{2/(\nu+1)}} \right]^\nu \\ &= n^{2\nu/(\nu+1)} \left[1 + \sum_{k=1}^{\nu} \binom{\nu}{k} \frac{\sigma^k}{n^{2k/(\nu+1)}} \right]. \end{aligned}$$

Hence, there exists a number M such that the following estimate holds on Γ_n :

$$|\lambda g(\lambda)| \leq M |\lambda|^\nu \leq M n^{2\nu/(\nu+1)} \left[1 + \sum_{k=1}^{\nu} \binom{\nu}{k} \frac{\sigma^k}{n^{2k/(\nu+1)}} \right]. \quad (26)$$

Note that the sum converges to zero for $n \rightarrow +\infty$. We now consider $f_n(\lambda)$ on Γ_n . We recall $\zeta_n^{(\nu+1)} = -n^2$.

$$\begin{aligned} |f_n(\lambda)| &= \left| [\zeta_n + \sigma e^{i\omega}]^{\nu+1} + n^2 \right| = n^2 \left| 1 - \left[1 + \frac{\sigma e^{i\omega}}{\zeta_n} \right]^{\nu+1} \right| \\ &= n^2 \left| \sum_{k=1}^{\nu+1} \binom{\nu+1}{k} \frac{[\sigma e^{i\omega}]^k}{\zeta_n^k} \right| \\ &\geq n^{2\nu/(\nu+1)} \sigma (\nu+1) \left[1 - \left| \sum_{k=2}^{\nu+1} \binom{\nu+1}{k} \frac{[\sigma e^{i\omega}]^k}{\sigma (\nu+1) \zeta_n^{k-1}} \right| \right]. \quad (27) \end{aligned}$$

The sum converges to 0 for $n \rightarrow +\infty$, uniformly for $\omega \in [0, 2\pi]$. We compare with (26). We choose the number σ so to have

$$\sigma > \frac{4M}{\nu+1}.$$

With this choice the following inequality holds for every n :

$$(\nu+1)\sigma n^{2\nu/(\nu+1)} > 4M n^{2\nu/(\nu+1)}.$$

The value of σ is now fixed.

We observe that the disks are contained in \mathcal{S} for every large enough index n and that the sum in (26) tends to zero so that there exists N' such that for every $n > N'$ and $\lambda \in \Gamma_n$ we have

$$\begin{aligned} &(\nu+1)\sigma n^{2\nu/(\nu+1)} \\ &> 3M n^{2\nu/(\nu+1)} \left\{ 1 + \sum_{k=1}^{\nu} \binom{\nu}{k} \left[\frac{\sigma^k}{n^{2k/(\nu+1)}} \right] \right\} \geq 3|\lambda g(\lambda)|. \quad (28) \end{aligned}$$

We consider (27). The fact that the sum converges to zero uniformly for $\omega \in [0, 2\pi]$ shows the existence of $N_0 > N'$ such that, for $n > N_0$ and $\lambda \in \Gamma_n$ we have

$$|f(\lambda)| \geq \frac{3}{2} |\lambda g(\lambda)|. \quad (29)$$

Rouché Theorem shows that $\lambda + n^2 G(\lambda)$ has a zero μ_n in every disk bounded by Γ_n provided that $n > N_0$ and this implies $\sum 1/|\mu_n|^\gamma = +\infty$, so that interpolation is impossible. ■

Remark 5 We observe:

- The previous results are in sharp contrast with the picture we have when $\theta = x \in \mathbb{R}^n$. Note that, in spite of the fact that null controllability is impossible, null controllability of the projection on a finite dimensional eigenspace of A might be achievable. But, when $\theta(t)$ evolves in an infinite dimensional space, the control which cancels, for example, the first components of θ disturbs the remaining components so that we don't have controllability to the rest. The case treated in examples 1 and 2 can be seen as a special instance of this fact since in those examples the eigenspaces of A span the space. This shows that the obstruction to controllability to the rest is the combination of infinite memory *and* infinite dimensionality.

Note a special and important case. Let θ evolve in an infinite dimensional space and let $\xi = \phi_k$ be an eigenvector of the operator A . The corresponding moment problem cannot be solved. We explicitly noted this in the proof of Lemma 2. So, ϕ_k cannot be controlled to the rest. In the case of systems without memory, controllability to the rest of initial data which are eigenvectors (or linear combination of finitely many eigenvectors) is called “spectral controllability” see for example [2, 11]. We note that the term “spectral controllability” has also a different meaning, see for example [14, 15] while in [1, p. 162] the reachability of the eigenvectors of A is termed “M-controllability”.

- Theorem 4 can be applied in particular if $\alpha = 0$ and $\hat{N}(\lambda) = 1/d(\lambda)$ where $d(\lambda)$ is a polynomial of degree at least 2, as in Example 2.
- The conclusion of Theorem 4 can be achieved using a more advanced argument, based on the Blaschke necessary and sufficient condition for interpolation in the Hardy spaces $\mathcal{H}^2(\Re z > 0)$ at the points μ_n , $\Re \mu_n >$

0 (see Theorem [1, p. 55]): the condition is that

$$\sum_{n=1}^{\infty} \frac{\Re \mu_n}{1 + |\mu_n|^2} < +\infty \quad (30)$$

If the estimate $|g(\lambda)| < M|\lambda|^{\nu-1}$ holds in a sector containing a chain of roots ξ_n in the right half plane, then the estimates given in theorem 4 for the sequence $\{\mu_n\}$ shows that the series (30) is not convergent (in fact, in the sector $-\pi/2 < a < \arg z < b < \pi/2$ the series in (30) and in (17) diverge simultaneously) so that interpolation is impossible.

- The assumption on the sector containing one of the rays $\arg z = \frac{2\pi k + \pi}{\nu+1}$ is satisfied if it happens that the required inequality on the function $g(\lambda)$ holds in an angle larger than $2\pi/(\nu+1)$ radians, since $2\pi/(\nu+1)$ is the angle among the rays. ■

The case that the kernel $N(t)$ is of Abel type,

$$N(t) = \frac{1}{\Gamma(1-\gamma)} t^{-\gamma} \quad \text{i.e.} \quad \hat{N}(\lambda) = \frac{1}{\lambda^{1-\gamma}}$$

with $0 < \gamma < 1$ and $\Gamma(\lambda)$ the Euler Γ -function, is important and has been often studied (see for example [7, 8]).

We have:

Theorem 6 *If $\hat{N}(\lambda) = 1/\lambda^{1-\gamma}$, $\gamma \in (0, 1)$, then controllability to the rest is not achievable.*

Proof. In fact, let $u \in \mathcal{PW}_+$ and let $\theta(t)$ be the corresponding solution to the Problem **i**). Then we have

$$\hat{\theta}_n(\lambda) = \frac{1}{\lambda + n^2[\alpha + 1/\lambda^\sigma]} \left\{ \xi_n + \frac{n}{\sqrt{\pi/2}} \left(\alpha + \frac{1}{\lambda^\sigma} \right) \hat{u}(\lambda) \right\}$$

where $\sigma = 1 - \gamma$. We prove that if γ , i.e. σ is not an integer, and $\hat{\theta}_n(\lambda)$ is analytical at the origin, then \hat{u} is singular there. For this, we replace $\lambda = \rho e^{i\omega}$ (ρ "small"). If $\hat{\theta}(\lambda)$ is regular at $\lambda = 0$ then we have $\lim_{\omega \rightarrow 0} \hat{\theta}(\rho e^{i\omega}) = \lim_{\omega \rightarrow 2\pi} \hat{\theta}(\rho e^{i\omega})$ and the same for $\hat{u}(\rho e^{i\omega})$. So, the following equality must hold:

$$\begin{aligned} & \frac{1}{\rho + n^2[\alpha + 1/\rho^\sigma]} \left[\xi_n + \frac{n\sqrt{2}}{\sqrt{\pi}} \left(\alpha + \frac{1}{\rho^\sigma} \right) \hat{u}(\rho) \right] \\ &= \frac{1}{\rho + n^2[\alpha + 1/(\rho^\sigma e^{2\pi\sigma i})]} \left[\xi_n + \frac{n\sqrt{2}}{\sqrt{\pi}} \left(\alpha + \frac{1}{\rho^\sigma e^{2\pi\sigma i}} \right) \hat{u}(\rho) \right]. \end{aligned}$$

We reduce to the same denominator and we find the equality

$$n(1 - e^{-2\pi\sigma i}) \frac{\xi_n}{\rho} = \sqrt{\frac{2}{\pi}} (1 - e^{-2\pi\sigma i}) \hat{u}(\rho).$$

This equality must hold for every index n . If γ , i.e. σ , is not an integer then $(1 - e^{2\pi\sigma i}) \neq 0$ and we see that $\hat{u}(\rho)$ is unbounded for $\rho \rightarrow 0+$ if $\xi_n \neq 0$ for one index n . This contradiction shows that controllability to the rest can be achieved only if the initial condition ξ is $\xi = 0$.

The case of Problem ii) can be treated similarly. ■

Remark 7 We note:

- The fact that θ evolves in an infinite dimensional space has not been used in the proof of Theorem 6, which holds also if $\theta = x \in \mathbb{R}^n$ (and the control entering in the equation, of course). For example, controllability to the rest is impossible even for the system

$$\dot{x}(t) = \int_0^t \frac{1}{(t-s)^\gamma} x(s) ds + u(t), \quad x \in \mathbb{R}.$$

- The results in this section have been stated for boundary control systems, and Dirichlet control. If the control is in the Neumann condition than we get similar interpolation problems and completely analogous negative results. The difference is that for Neumann control we obtain, instead of (10),

$$\hat{\theta}_n(\lambda) = -\frac{1}{\lambda + n^2 G(\lambda)} (\xi_n + \phi_n(0) G(\lambda) \hat{u}(\lambda)).$$

Here $\phi_n(t)$ are the eigenfunctions of the Neumann problem, so that $\phi_n(0) \neq 0$.

- Needless to say, the negative results in this section hold also if the interval $(0, \pi)$ is replaced by any interval (a, b) . ■

4.2 Case iii): control distributed on a subdomain

The negative results in the previous section have been independent of any existence/uniqueness and regularity theorem for the solutions. Instead, the results in this section do depend on existence theorems and regularity of the solutions.

In this section the system is acted upon by a distributed control which is supported on an interval $[\beta, \gamma] \subseteq (0, \pi)$. Hence, there is an interval $[a, b] \subseteq$

$[0, \pi]$ which does not intersect $[\beta, \gamma]$. Clearly, we can choose either $a = 0$ or $b = \pi$. We consider the case $b = \pi$. The case $a = 0$ is treated analogously. We shall consider the restriction of the solution $\theta(t, \cdot)$ of Equation (4) to the interval $(a, b) = (a, \pi)$. This function solves the equation

$$\theta_t = \alpha \theta_{xx} + \int_0^t N(t-s) \theta_{xx}(s) ds, \quad x \in (a, b), \quad t > 0 \quad (31)$$

with initial condition $\theta(0, x) = \xi(x)$ for $x \in (a, b)$ and boundary conditions

$$\theta(t, \pi) = 0, \quad \theta(t, a+) = \theta(t, a-),$$

provided that we can give a meaning to $\theta(t, a-)$.

In this case, $\theta(t, a-) = v(t)$ acts as a boundary control and we know that if the kernel has the properties in the previous section then controllability to the rest of the restriction of $\theta(t, \cdot)$ to (a, b) , hence also of $\theta(t, x)$, $x \in (0, \pi)$, is impossible.

We are going to give conditions under which the previous argument makes sense:

Lemma 8 *Let $u \in L_{\text{loc}}^2([0, +\infty); L^2(0, \pi))$ and let $\xi \in H_0^1(0, \pi) = \text{dom}(-A)^{1/2}$. We have that $\theta(t, \cdot) \in H_0^1(0, \pi)$ for every $t \geq 0$, provided that*

- a) *either $\alpha > 0$ and $N(t) \in H^1(0, T)$ for every $T > 0$*
- b) *or $\alpha = 0$, $N(t) \in H^2(0, T)$ for every $T > 0$ and $N(0) > 0$.*

Proof. We fix any $T > 0$. We consider case **a)** first. In this case $\alpha > 0$ and we can assume $\alpha = 1$ without restriction. Then (see [4, Section 2]), $\theta(t) = \theta(t, x)$ solves

$$\begin{aligned} \theta(t) = & e^{At} \xi - \int_0^t N(t-r) \theta(r) dr + \\ & \int_0^t e^{A(t-s)} \left[N(0) \theta(s) + \int_0^s N'(s-r) \theta(r) dr \right] ds + \int_0^t e^{A(t-s)} B u(s) ds. \end{aligned} \quad (32)$$

Here e^{At} is the holomorphic semigroup generated by the operator A in (6) and B is the operator from \mathbb{R} to $L^2(0, \pi)$, $Bu = b(x)u$. The function $b(x)$ is square integrable, with support in $[0, \pi] - [a, b]$. It is known that the transformation

$$u \rightarrow \int_0^t (-A)^{1/2} e^{A(t-s)} B u(s) ds$$

is continuous from $L^2(0, T; L^2(0, \pi))$ to itself (see [12] for an even more general case). We consider now the following Volterra integral equation:

$$\begin{aligned} y(t) &= e^{At}(-A)^{1/2}\xi + \int_0^t (-A)^{1/2}e^{A(t-s)}Bu(s) \, ds - \int_0^t N(t-r)y(r) \, dr \\ &+ \int_0^t e^{A(t-s)} \left[y(s) + \int_0^s N'(s-r)y(r) \, dr \right] \, ds. \end{aligned}$$

Both this equation and Eq. (32) are solvable and have a unique solution. Hence it must be $\theta(t) = (-A)^{-1/2}y(t) \in \text{dom}(-A)^{1/2} = H_0^1(0, \pi)$, as wanted.

Now we compute the inner product of $\theta(t)$ with $\phi \in \text{dom} A$. We use (32) and we see that $\langle \theta(t), \phi \rangle$ solves a scalar Volterra integral equation with smooth coefficients, hence it is differentiable and the derivative is

$$\begin{aligned} \frac{d}{dt} \langle \theta(t), \phi \rangle &= \langle e^{At}\xi, A\phi \rangle + \langle Bu(t), \phi \rangle + \int_0^t \langle e^{A(t-s)}Bu(s), A\phi \rangle \, ds \\ &+ \int_0^t \langle N(0)\theta(s) + \int_0^s N'(s-r)\theta(r) \, dr, e^{A(t-s)}A\phi \rangle \, ds. \end{aligned}$$

Simple manipulations show that

$$\frac{d}{dt} \langle \theta(t), \phi \rangle = \langle \theta(t), A\phi \rangle + \int_0^t N(t-r)\langle \theta(r), A\phi \rangle \, dr + \langle Bu(t), \phi \rangle. \quad (33)$$

We can consider in particular any $\phi \in H^2(0, \pi)$ which is zero for $t \notin (a, b) = (a, \pi)$. In particular, ϕ any eigenvector ϕ_n of the operator $\tilde{A} = \partial^2/\partial x^2$, with domain $H^2(a, \pi) \cap H_1^0(a, \pi)$. Using equality (5) with now $\phi'(0)$ replaced by $\phi'(a)$ and (33), we see that $\langle \theta(t), \phi_n \rangle$ verifies equation (9), written on the interval (a, π) , so that the negative results in Section 4.1 can be applied, as we wanted to show.

The proof in case **b**) is analogous, but depends on the use of cosine operators. In this case, it is not restrictive to assume $N(0) = 1$. The function $\theta(t)$ solves the Volterra integral equation

$$\theta(t) = \left\{ R_+(t)\xi + \int_0^t R_+(t-s)Bu(s) \, ds \right\} + \int_0^t L(t-s)\theta(s) \, ds, \quad (34)$$

where $R_+(t)$ is the cosine operator generated by the operator A , see [20, formulas 13 and 14].

The function $L(t)$ is now

$$L(t)\theta = R_+(t)N'(0)\theta - N'(t)\theta + \int_0^t R_+(t-s)N''(s)\theta \, ds.$$

The function

$$x(t) = \int_0^t R_+(t-s)Bu(s) ds$$

solves $x_{tt} = x_{ss} + Bu(t)$ (and zero initial and boundary conditions) so that $x(t) \in C(0, T; H_0^1(0, \pi)) = C(0, T; \text{dom}(-A)^{1/2})$, see [6, p. 34-35].

Let $\xi \in H_0^1(0, \pi) = \text{dom}(-A)^{1/2}$. We consider the Volterra integral equation

$$y(t) = \left\{ R_+(t)(-A)^{1/2}\xi + (-A)^{1/2} \int_0^t R_+(t-s)Bu(s) ds \right\} + \int_0^t L(t-s)y(s) ds$$

and we proceed as above to show $\theta(t) = (-A)^{-1/2}y(t)$ and to apply the negative results in Section 4.1. ■

In conclusion:

Theorem 9 *Let the assumptions in Lemma 8 hold and let $u(t, x) = b(x)v(t, x)$. Let the input operator $b(x)$ have support in a compact subset of $(0, \pi)$. Let furthermore the assumptions of one of the theorems in Section 4.1 hold. Then, the initial condition ξ cannot be controlled to the rest.*

Note that the initial conditions we considered in the proof of Theorem 9 belong to $H_0^1(0, \pi)$ but the obstructions to interpolation applies to this set of initial conditions as well, see Lemma 2.

5 Conclusion

In this paper we presented negative results on the controllability to the rest for the heat equation with memory, which shows that the cases in which controllability to the rest is achievable must be very particular: cases in which the Laplace transform of the kernel does not have neither branch points nor zeros or multiple poles. Even more, we proved that if $N(t)$ has a rational Laplace transform then $\hat{N}(\lambda)$ must have the form $a/(\lambda + b)$. I.e., in this case the heat equation with memory must be an “integrated” form of the wave type equation: if $b \neq 0$ the wave equation with an additional term.

See [16] for a different kind of negative results due to the presence of infinite memory.

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